Exam 1 Review

exam format: 8 questions (12 pts each, 4 bonus)

"hybrid multiple-choice"

→ 25% on answer

→ 75% on work

4" x 6" note card

→ handwritten

→ no sharing

→ must be 2D

Linear

nonlinear

Laplace

$$\vec{x}' = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \vec{x}$$

$\lambda = 4, 4$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

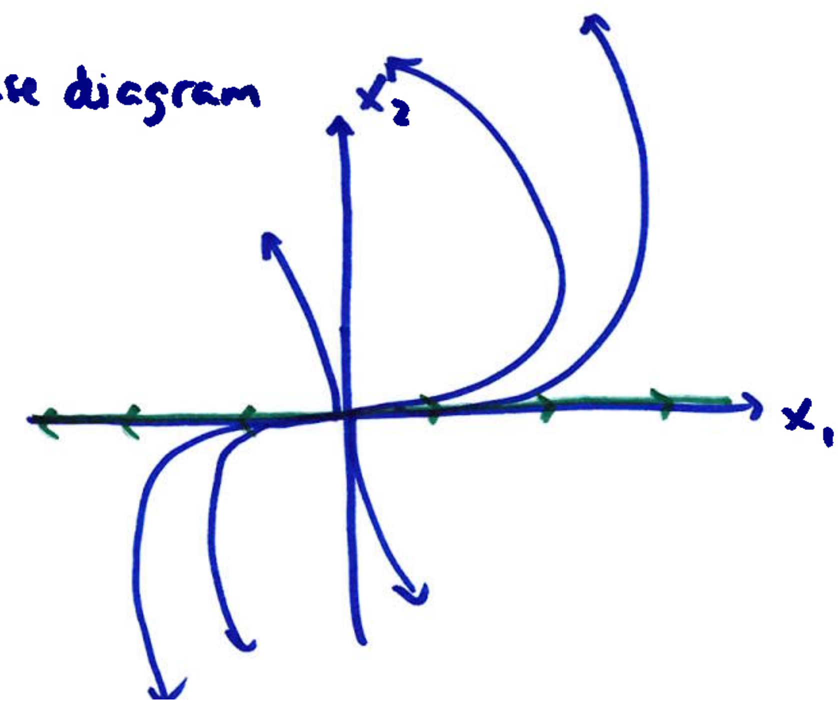
missing one eigenvector

generalized eigenvector: $(A - \lambda I)\vec{u} = \vec{v}$

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

solution: $\vec{x} = c_1 e^{4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{4t} \left(t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

phase diagram



origin is a nodal source (improper)

$\vec{x}' = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix} \vec{x}$

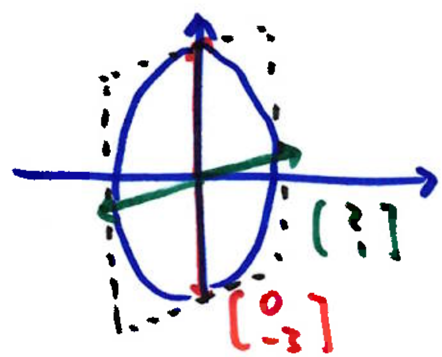
$\lambda = 3i,\ -3i$

$\vec{v} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix},\ \begin{bmatrix} 2 \\ 1+3i \end{bmatrix}$

sketch phase diagram

λ's are purely imaginary: ovals

orientation: $\begin{bmatrix} 2 \\ 1-3i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -3 \end{bmatrix}$



direction: $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 5 & -1 \end{bmatrix} \vec{x}$ pick $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{x}' = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

right, up

counterclockwise

ie origin is a center

stable

nonlinear systems

$$x' = x(1-y)$$
$$y' = y(x-3)$$

predator-prey

$$x' = x - xy$$
$$y' = -3y + xy$$

Critical pts: $x'=0$ AND $y'=0$

$$x'=0 \rightarrow x=0, \; y=1$$
$$y'=0 \rightarrow y=0, \; x=3$$

$$(0,0), (3,1)$$

Jacobian matrix

$$x' = f(x,y)$$
$$y' = g(x,y)$$

$$J(x,y) = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

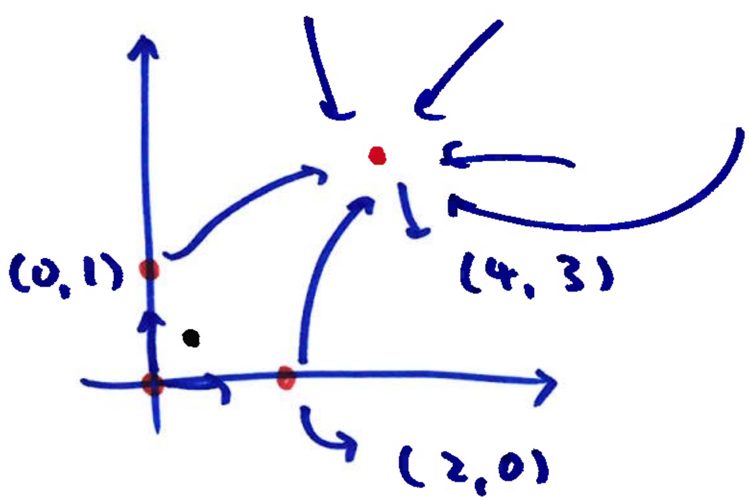
here, $J = \begin{bmatrix} 1-y & -x \\ y & x-3 \end{bmatrix}$

$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$ $\lambda = 1, -3$

(0, 0) is a saddle pt
unstable
solutions that start near (0,0) go away from (0, 0)

$J(3,1) = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix}$ λ's are purely imaginary

(3, 1) is a center
solutions that start near (3, 1) orbit (3, 1)



has higher pops than if only one species is around

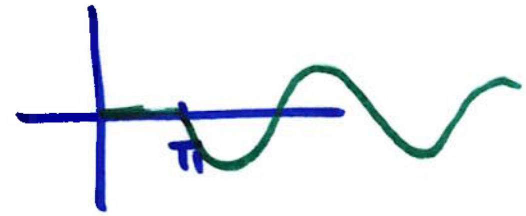
both helped by the other

→ cooperation

unit step and impulse

$$f(t) = \begin{cases} 0 & 0 < t < \pi \\ \sin(t) & t \geq \pi \end{cases}$$

$F(s) = ?$



$= u_{\pi}(t) \cdot \sin(t)$

shift LEFT by π : $t \to t+\pi$

$$F(s) = e^{-\pi s} \mathcal{L}\{\sin(t+\pi)\}$$

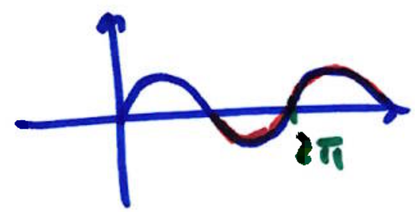
$$= e^{-\pi s} \mathcal{L}\{-\sin(t)\}$$

$$= e^{-\pi s} \cdot \frac{-1}{s^2+1}$$

$\sin(t)$ has period of 2π

$\sin(t+\pi)$ shifted by half period

$= -\sin(t)$



convolution: $\int_0^t f(\tau)\, g(t-\tau)\, d\tau = \int_0^t f(t-\tau)\, g(\tau)\, d\tau = f * g$

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\} = FG$$

$$\int_0^t \underbrace{(t-\tau)^2}_{f(t-\tau)} \underbrace{\delta(\tau-3)}_{g(\tau)}\, d\tau = ?$$

in S-domain: $f(t) = t^2, \quad g(t) = \delta(t-3)$

$$f(t) = t^2 \downarrow \frac{2}{s^3}, \qquad g(t) = \delta(t-3) \downarrow e^{-3s}$$

$$e^{-3s}\left(\frac{2}{s^3}\right) \to t^2$$

shift RIGHT by 3: $t \to t-3$

back to t: $u_3(t)\cdot(t-3)^2$

$$y'' + 4y = \delta(t-\pi) \qquad y(0) = y'(0) = 0$$

$$s^2 Y + 4Y = e^{-\pi s}$$

$$(s^2+4)Y = e^{-\pi s}$$

$$Y = e^{-\pi s} \cdot \frac{1}{s^2+4} \qquad \frac{1}{2} \cdot \frac{2}{s^2+4} = \frac{1}{2}\sin(2t)$$

shift: $t \to t - \pi$

$$y = u_\pi(t) \cdot \frac{1}{2}\sin(2(t-\pi))$$